Journal of Engineering Mathematics **30:** 19–24, 1996.

A general theorem on the motion of a fluid with friction and a few results derived from it *

§1. The equations of motion for an incompressible fluid with friction can be written as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,, \tag{1}$$

$$\left.\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) &= \mathbf{X}+\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) &= \mathbf{Y}+\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) &= \mathbf{Z}+\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right)
\end{aligned}\right\}. \tag{2}$$

Here ρ represents the density, $u,\ v,\ w$ the velocity components, **X**, **Y**, **Z** the components of the external force per unit of volume and $X_x,\ X_y$, etc. the stress components on planes perpendicular to the coordinate axes.

If p is the pressure and μ the friction coefficient, the latter quantities are given by the equations

$$X_x=-p+2\mu\frac{\partial u}{\partial x},\quad \text{etc.}, \tag{3}$$

$$X_y=Y_x=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad \text{etc.} \tag{4}$$

§2. We assume that a given space τ, bounded by a surface σ (or more than one surface), is entirely filled with fluid. The space and surface elements are denoted by $d\tau$ and $d\sigma$, respectively, and the integrals appearing in the equations which follow extend over the whole of the space τ or the complete surface σ. The latter might be the interface between the fluid and a solid, or else an arbitrary surface within the fluid mass. We shall denote by n the outward normal with respect to the space τ, by $\alpha,\ \beta,\ \gamma$ the angles between this normal and the coordinate axes and by $X_n,\ Y_n,\ Z_n$ the stress components on the surface, so that

$$X_n=X_x\cos\alpha+X_y\cos\beta+X_z\cos\gamma\,,\quad \text{etc.} \tag{5}$$

Now, the theorem in question can be obtained if we assume *two* states of motion, both of which satisfy the equations of motion. For the *first* state of motion we shall use the symbols introduced above, for the *second* the same symbols with primes whenever these are necessary.

* English translation of H.A. Lorentz, Eene algemeene stelling omtrent de beweging eener vloeistof met wrijving en eenige daaruit afgeleide gevolgen. *Zittingsverslag van de Koninklijke Akademie van Wetenschappen te Amsterdam* 5 (1896) 168-175.

This second state satisfies equations which are identical with (1) – (5). There is no need to write these down separately, as we can denote these by (1′) – (5′).

§3. Let us now consider the integral

$$\Omega = \int (u' X_n + v' Y_n + w' Z_n)\, \mathrm{d}\sigma .$$

By making use of the relations (5) and realising that for an arbitrary function φ

$$\int \varphi \cos\alpha\, \mathrm{d}\sigma = \int \frac{\partial\varphi}{\partial x}\, \mathrm{d}\tau ,$$

$$\int \varphi \cos\beta\, \mathrm{d}\sigma = \int \frac{\partial\varphi}{\partial y}\, \mathrm{d}\tau , \quad \text{etc.,}$$

we find for that expression

$$\Omega = \int \left[\frac{\partial(u' X_x)}{\partial x} + \frac{\partial(u' X_y)}{\partial y} + \frac{\partial(u' X_z)}{\partial z} + \text{etc.}\right] \mathrm{d}x .$$

Here and later the word "etc." will be used to indicate that terms must be added, which follow from the expression as it has been written down, by applying twice a cyclic transposition of symbols.

We can now divide Ω in two parts. The first reads

$$\Omega_1 = \int \left[u' \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) + \text{etc.}\right] \mathrm{d}\tau ,$$

and the second

$$\Omega_2 = \int \left[X_x \frac{\partial u'}{\partial x} + X_y \frac{\partial u'}{\partial y} + X_z \frac{\partial u'}{\partial z} + \text{etc.}\right] \mathrm{d}\tau ,$$

or, upon substitution of (3) and (4) and having regard to (1′)

$$\Omega_2 = \mu \int \left[2 \frac{\partial u}{\partial x} \frac{\partial u'}{\partial x} + \text{etc.} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + \text{etc.}\right] \mathrm{d}\tau .$$

Since this expression does not change upon interchanging $u,\ v,\ w$ and $u',\ v',\ w'$, we will find the same result if we proceed from

$$\Omega' = \int (u X'_n + v Y'_n + w Z'_n)\, \mathrm{d}\sigma ,$$

and treating it in the same way as Ω.

Therefore, writing

$$\Omega'_1 = \int \left[u \left(\frac{\partial X'_x}{\partial x} + \frac{\partial X'_y}{\partial y} + \frac{\partial X'_z}{\partial z}\right) + \text{etc.}\right] \mathrm{d}\tau ,$$

we have

$$\Omega - \Omega' = \Omega_1 - \Omega'_1 .$$

Now we substitute in Ω_1 and Ω_1' the values of

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \quad \text{etc.},$$

which follow from the equations (2) and (2′).

Replacing

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \quad \text{etc.}$$

by

$$\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}, \quad \text{etc.},$$

which we are allowed to do because of (1) and (1′), we obtain the theorem

$$\int (u'X_n + v'Y_n + w'Z_n)\,\mathrm{d}\sigma - \int (uX_n' + vY_n' + wZ_n')\,\mathrm{d}\sigma =$$

$$= \rho \int \left[\left(u'\frac{\partial u}{\partial t} - u\frac{\partial u'}{\partial t}\right) + \text{etc.}\right] \mathrm{d}\tau - \int \left[(u'\mathbf{X} - u\mathbf{X}') + \text{etc.}\right] \mathrm{d}\tau +$$

$$+\rho \int \left[u'\left\{\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}\right\} - u\left\{\frac{\partial(u'^2)}{\partial x} + \frac{\partial(u'v')}{\partial y} + \frac{\partial(u'w')}{\partial z}\right\} + \text{etc.}\right] \mathrm{d}\tau. \tag{I}$$

§4. First, we shall apply this theorem assuming that the two states of motion are stationary, that neither the one nor the other are influenced by external forces and that all velocities are infinitesimally small. The equation is then transformed into

$$\int (u'X_n + v'Y_n + w'Z_n)\,\mathrm{d}\sigma - \int (uX_n' + vY_n' + wZ_n')\,\mathrm{d}\sigma = 0\,. \tag{II}$$

Apart from (1) and (1′), these two states of motion must now satisfy

$$-\frac{\partial p}{\partial x} + \mu\Delta u = 0\,, \quad \text{etc.} \tag{6}$$

and

$$-\frac{\partial p'}{\partial x} + \mu\Delta u' = 0 \quad \text{etc.} \tag{6′}$$

For the *first* of these states we select one which actually exists in one way or another in the space τ, with values of $u,\ v,\ w$ etc. which are everywhere finite and continuous, for the *second*, on the other hand, an imaginary state which we shall define as follows.

Let P be an arbitrary point in the space τ, B a small sphere with that particular point as its centre and with radius R, and let $u',\ v',\ w'$ be such as would be the case when the fluid surrounding B extended into infinity with

$$u' = c, \quad v' = 0, \quad w' = 0 \quad \text{(with } c \text{ an arbitrarily small constant)}$$

at the surface of B and with

$$u' = v' = w' = 0$$

at infinity.

Then we have, with P as the origin of the coordinate system and with r measuring distance from P,

$$u' = -\frac{1}{4}R^3c\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) + \frac{3}{4}Rc\left(\frac{x^2}{r^3} + \frac{1}{r}\right),$$

$$v' = -\frac{3}{4}R^3c\frac{xy}{r^5} + \frac{3}{4}Rc\frac{xy}{r^3},$$

$$w' = -\frac{3}{4}R^3c\frac{xz}{r^5} + \frac{3}{4}Rc\frac{xz}{r^3},$$

from which we may readily derive the values of the stress components.

Applying theorem (II) to the space between the spherical surface B and the surface σ, allowing R to tend to 0, we find

$$u_P = \frac{3}{4\pi}\int \frac{x}{r^5}(ux + vy + wz)(x\cos\alpha + y\cos\beta + z\cos\gamma)\,\mathrm{d}\sigma +$$
$$+\frac{1}{8\pi\mu}\int \left[\left(\frac{x^2}{r^3} + \frac{1}{r}\right)X_n + \frac{xy}{r^3}Y_n + \frac{xz}{r^3}Z_n\right]\mathrm{d}\sigma. \tag{7}$$

Here u_P denotes the velocity u at the point P; since there exist similar equations for v_P and w_P, we have answered the question of how the velocity at an arbitrary point in space depends upon the velocities and stress components at the bounding surface.

§5. One can use this result to determine how a given state of motion is "reflected" by a fixed plane surface along which the fluid cannot slide. To that end we shall first consider the following problem.

To find the relationship between two states of motion $(u_1,\ v_1,\ w_1,\ p_1)$ and $(u_2,\ v_2,\ w_2,\ p_2)$, both of which extend in the space on the positive side of the yz-plane and which satisfy the conditions

$$u_1 = u_2,\quad v_1 = -v_2,\quad w_1 = -w_2 \tag{8}$$

everywhere on this plane.

The solution is found by an application of (7) to both states of motion (such that the plane σ is allowed to coincide with the yz-plane) and the employment of several mathematical artifices which, for the sake of brevity, have to be omitted here. Finally, we find for each point in the space considered

$$\left.\begin{aligned}
u_2 &= u_1 - 2x\frac{\partial u_1}{\partial x} + \frac{x^2}{\mu}\frac{\partial p_1}{\partial x},\\
v_2 &= -v_1 - 2x\frac{\partial u_1}{\partial y} + \frac{x^2}{\mu}\frac{\partial p_1}{\partial y},\\
w_2 &= -w_1 - 2x\frac{\partial u_1}{\partial z} + \frac{x^2}{\mu}\frac{\partial p_1}{\partial z},\\
p_2 &= p_1 + 2x\frac{\partial p_1}{\partial x} - 4\mu\frac{\partial u_1}{\partial x}
\end{aligned}\right\}. \tag{9}$$

Indeed, it can be seen immediately that the first three equations are reduced to (8) for $x = 0$ and a direct calculation will show that $u_2,\ v_2,\ w_2,\ p_2$ satisfy the equations of motion if this is true for $u_1,\ v_1,\ w_1,\ p_1$.

As far as the "reflection" problem is concerned, we shall assume that the fixed wall coincides with the yz-plane and that the fluid will be on the positive side of the x-axis. We suppose that a state of motion M_0 is generated, for instance by maintaining continually a velocity field at a small closed surface inside the fluid or by applying continually external forces on part of the fluid, this state of motion being known in the absence of the fixed wall and with the fluid extending in the space beyond the yz-plane (where x is negative). Let $u_0,\ v_0,\ w_0$ be the velocities which exist on the yz-plane for this particular flow field.

Now one can always imagine a state M_1 in front of the yz-plane which, as far as the velocities are concerned, is the mirror image of what M_0 would be beyond the yz-plane. On this plane the velocities of this state will read

$$u_1 = -u_0, \quad v_1 = v_0, \quad w_1 = w_0 .$$

We can now use the equations (9) to derive a state M_2 from the state M_1 with velocities $u_2,\ v_2,\ w_2$ satisfying

$$u_0 + u_2 = 0, \quad v_0 + v_2 = 0, \quad w_0 + w_2 = 0 ,$$

on the yz-plane.

Thus, in the presence of the fixed wall, M_2 is the state of motion which can co-exist with M_0.

§6. If it were possible (§4) for us to generate a state of motion with infinitesimally small velocities $u',\ v',\ w'$, which would satisfy

$$u' = c, \quad v' = 0, \quad w' = 0$$

on the surface of an infinitesimally small sphere B and

$$u' = v' = w' = 0$$

at the surface σ, then we would obtain from (II)

$$u_P = -\frac{1}{6\pi\mu c R}\int (uX'_n + vY'_n + wZ'_n)\,\mathrm{d}\sigma .$$

This would solve the problem of finding the velocities $u,\ v,\ w$ at an arbitrary interior point when arbitrary velocities are given at the interface σ.

§7. We shall now assume that a solid body L is positioned within the space enclosed by the surface σ and select for $(u,\ v,\ w)$ in (I) a state of motion M which is admissible when this body is maintained in a state of rest or has a given motion.[1] We select for $(u',\ v',\ w')$ a stationary situation with infinitesimal velocities and without external forces, with

$$u' = c, \ v' = 0, \ w' = 0$$

[1] However, it must be possible for this motion to exist in a stationary situation (continual rotation of an axisymmetric body about its axis).

on the surface of L and

$$u' = v' = w' = 0$$

on σ. Each surface integral of (I) now splits up into an integral on the interface σ and one on the surface Σ of L. Since the first integral in (I), taken on Σ, is identical with the net force Ξ which the body L experiences in the direction of the x-axis as a result of the motion M, when multiplied by $-c$, we obtain

$$\Xi = -\frac{1}{c}\int (uX_n' + vY_n' + wZ_n')\,\mathrm{d}\Sigma - \frac{1}{c}\int (uX_n' + vY_n' + wZ_n')\,\mathrm{d}\sigma +$$

$$+\frac{1}{c}\int (u'\mathbf{X} + v'\mathbf{Y} + w'\mathbf{Z})\,\mathrm{d}\tau - \frac{\rho}{c}\int \left[u'\left\{\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z}\right\} + \text{etc.}\right]\mathrm{d}\tau\,.$$

If the state of motion M does not involve external forces and if we are allowed to disregard the second-order terms, then the expression on the right is reduced to the first two terms; it is then possible to determine the force acting on L, as long as the velocities $u,\ v,\ w$ are known on the surfaces Σ and σ. One more term is involved when external forces $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are present with the velocities still being infinitesimal, but, still, it is not necessary to know $u,\ v$ and w at every point in space. This becomes necessary only when the second-order velocity terms are included, but it should be clear from the nature of the last term that we may substitute in it the values obtained from the equations of motion which are valid for infinitesimal velocities, that is, as long as we disregard terms which are of an even higher order.

This formula has been derived for a certain application, the study of which we have not yet fully completed.

Translated from Dutch by H.K. Kuiken (1995)